

A note on the best attainable rates of convergence for estimates of the shape parameter of regular variation

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Abstract

Hall and Welsh gave in 1984 the lowest bound so far to rates of convergence for estimates of the shape parameter of regular variation. We show that this bound can be improved.

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Hall and Welsh (hereafter HW) gave in 1984 a first lower bound of the accuracy of tail index estimation for a large class of distributions. Since then, this result has been a reference to evaluate rates of convergence for other estimators of this parameter, and has motivated extensions of it for other classes of distributions (say e.g. [1], [2], [5], [6], and [7]).

Let F be a differentiable distribution function (df) defined on the positive half-line such that, for positive constants α , β , C and A ,

$$F'(x) = C\alpha x^{\alpha-1}(1 + r(x)) \quad \text{where} \quad |r(x)| \leq Ax^\beta, \quad (1)$$

as $x \rightarrow 0^+$.

Considering this type of dfs, HW [4] showed in 1984 that no estimator of α converges at a faster rate than $n^{-\beta/(2\beta+\alpha)}$ on certain neighborhoods of Pareto distributions (see e.g. [7] or [2]). More precisely, these authors defined classes $\mathcal{D} = \mathcal{D}(\alpha_0, C_0, \epsilon, \rho, A)$ of dfs F satisfying (1) and, in addition, $|\alpha - \alpha_0| \leq \epsilon$, $|C - C_0| \leq \epsilon$ and $\rho = \beta/\alpha$ for some given positive constants α_0 , C_0 , ϵ and A . Let $\beta_0 = \rho\alpha_0$. Then it was shown that (see Theorem 1 in [4]), if α_n is an estimator of α , constructed out of a random n -sample X_1, \dots, X_n , satisfying

$$\lim_{n \rightarrow \infty} \inf_{F \in \mathcal{D}} P(|\alpha_n - \alpha| \leq a_n) = 1, \quad (2)$$

then

$$\lim_{n \rightarrow \infty} n^{\beta_0/(2\beta_0+\alpha_0)} a_n = \infty.$$

We find that the proof of this result, developed by HW, allows one, after some adequate modifications, to also prove

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Theorem 1. Suppose that for some α_0, C_0, ϵ and ρ , we have (2) for all $A > 0$. Then, for all $\nu \geq \beta_0/(2\beta_0 + \alpha_0)$,

$$\lim_{n \rightarrow \infty} n^\nu a_n = \infty.$$

This means that $n^{-\beta/(2\beta+\alpha)} = n^{-\beta_0/(2\beta_0+\alpha_0)} = n^{-\rho/(2\rho+1)}$ because of $\rho = \beta/\alpha = \beta_0/\alpha_0$, is no longer a lower bound to convergence rates for estimators of shape parameters in distributions with regularly varying tails, as claimed by Theorem 1 given in [4].

The proof of Theorem 1 is the same given by HW to prove Theorem 1 in [4], but redefining conveniently two parameters. We present these redefinitions and show how with these changes the original proof can still be applied.

In order to have a self-contained paper, we copy almost all of the proof of Theorem 1 in [4]. The main changes in that proof are pointed out.

Let $\nu \geq \beta_0/(2\beta_0 + \alpha_0)$.

For proving Theorem 1 in [4], HW started constructing two densities f_0 and f_1 , the first governed by fixed parameters α_0, C_0 and the second by varying parameters α_1, C_1, C_2 , where $\alpha_1 = \alpha_0 + \tilde{\gamma}$, $\tilde{\gamma} = \lambda n^{-\nu}$, $\lambda > 0$, $\beta_1 = \rho\alpha_1$ and both $C_1, C_2 \rightarrow C_0$ as $n \rightarrow \infty$.

Here we point out that we use $\tilde{\gamma}$ instead of γ . HW used γ in the proof of Theorem 1 in [4], where these authors defined it as $\gamma = \lambda n^{-\beta_1/(2\beta_1+\alpha_1)}$.

Specifically, HW defined

$$f_0(x) = C_0 \alpha_0 x^{\alpha_0-1}, \quad 0 \leq x \leq C_0^{-1/\alpha_0},$$

and

$$f_1(x) = \begin{cases} C_1 \alpha_1 x^{\alpha_1-1} + \Delta(x), & 0 \leq x \leq \tilde{\delta} \\ C_2 \alpha_0 x^{\alpha_0-1}, & \tilde{\delta} < x \leq C_0^{-1/\alpha_0}. \end{cases}$$

where $\tilde{\delta} = n^{-\nu/\beta_1}$, $k = \alpha_1 + \beta_1 - 1$ and

$$\Delta(x) = \begin{cases} x^k, & 0 < x \leq \tilde{\delta}/4 \\ (\tilde{\delta}/2 - x)^k, & \tilde{\delta}/4 < x \leq \tilde{\delta}/2 \\ -(x - \tilde{\delta}/2)^k, & \tilde{\delta}/2 < x \leq 3\tilde{\delta}/4 \\ -(\tilde{\delta} - x)^k, & 3\tilde{\delta}/4 < x \leq \tilde{\delta}. \end{cases}$$

Here we point out that we use $\tilde{\delta}$ instead of δ . HW used δ in the proof of Theorem 1 in [4], where these authors defined it as $\delta = n^{-1/(2\beta_1+\alpha_1)}$.

One can note that $\Delta(x)$ is continuous on $[0; \tilde{\delta}]$, that $\Delta(0) = \Delta(\tilde{\delta}) = 0$ and

$$\int_0^{\tilde{\delta}} \Delta(x) dx = 0.$$

HW chose the constants C_1, C_2 so that for large n , f_1 is a proper, continuous density on $[0; C_0^{-1/\alpha_0}]$; that is,

$$C_1 \alpha_1 \tilde{\delta}^{\alpha_1} = C_2 \alpha_0 \tilde{\delta}^{\alpha_0} \quad (3)$$

and

$$C_1 \tilde{\delta}^{\alpha_1} + C_2 (C_0^{-1} - \tilde{\delta}^{\alpha_0}) = 1. \quad (4)$$

Note that from (3)

$$\lim_{n \rightarrow \infty} (C_1 - C_2) = \lim_{n \rightarrow \infty} C_2 \left(\frac{\alpha_0}{\alpha_1} \tilde{\delta}^{-\tilde{\gamma}} - 1 \right) = 0$$

and from (3) and (4)

$$C_2 - C_0 = C_0 \left(C_2 \tilde{\delta}^{\alpha_0} - C_1 \tilde{\delta}^{\alpha_1} \right) = C_0 \left(C_1 \frac{\alpha_1}{\alpha_0} \tilde{\delta}^{\alpha_1} - C_1 \tilde{\delta}^{\alpha_1} \right) = \frac{C_0 C_1}{\alpha_0} \tilde{\gamma} \tilde{\delta}^{\alpha_1}, \quad (5)$$

which gives

$$\lim_{n \rightarrow \infty} (C_2 - C_0) = \lim_{n \rightarrow \infty} \frac{C_0 C_1}{\alpha_0} \tilde{\delta}^{\alpha_1} \tilde{\gamma} = 0.$$

This guarantees that $C_1, C_2 \rightarrow C_0$ as $n \rightarrow \infty$, as required for C_1 and C_2 .

Then, the proof given by HW consisted initially of showing that

$$\int_0^{C_0^{-1/\alpha_0}} (f_0(x) - f_1(x))^{-2} f_0(x) dx = O(n^{-1}) \quad (6)$$

as $n \rightarrow \infty$, and for all large n ,

$$f_1 \in \mathcal{D}(\alpha_0, C_0, \epsilon, \rho, A). \quad (7)$$

Note that trivially $f_0 \in \mathcal{D}$. The symbol K denotes a positive generic constant.

By (5), as $n \rightarrow \infty$,

$$|C_2 - C_0| = O(\tilde{\gamma} \tilde{\delta}^{\alpha_1}). \quad (8)$$

We also have, as $n \rightarrow \infty$,

$$\begin{aligned} & (2\alpha_1 - \alpha_0) \int_0^{\tilde{\delta}} (C_0 \alpha_0 x^{\alpha_0-1} - C_1 \alpha_1 x^{\alpha_1-1})^2 (x^{\alpha_0-1})^{-1} dx \\ &= (2\tilde{\gamma} + \alpha_0) \int_0^{\tilde{\delta}} (C_0^2 \alpha_0^2 x^{\alpha_0-1} - 2C_0 C_1 \alpha_0 \alpha_1 x^{\alpha_1-1} + C_1^2 \alpha_1^2 x^{2\alpha_1-\alpha_0-1}) dx \\ &= O(\tilde{\delta}^{\alpha_1} (C_0 - C_1 \tilde{\delta}^{\tilde{\gamma}})^2 + \tilde{\gamma}^2 \tilde{\delta}^{\alpha_1}); \end{aligned} \quad (9)$$

using (5)

$$\begin{aligned} C_0 - C_1 \tilde{\delta}^{\tilde{\gamma}} &= C_0 - C_2 + \tilde{\delta}^{-\alpha_0} (C_2 C_0^{-1} - 1) = (C_0 - C_2) (1 - C_0^{-1} \tilde{\delta}^{-\alpha_0}) \\ &= -\frac{C_0 C_1}{\alpha_0} \tilde{\gamma} \tilde{\delta}^{\alpha_1} (1 - C_0^{-1} \tilde{\delta}^{-\alpha_0}) = \frac{C_1}{\alpha_0} \tilde{\gamma} (\tilde{\delta}^{\tilde{\gamma}} - C_0 \tilde{\delta}^{\alpha_1}) = O(\tilde{\gamma}); \end{aligned} \quad (10)$$

and

$$\int_0^{\tilde{\delta}} (\Delta(x))^2 x^{-\alpha_0+1} dx \leq K \int_0^{\delta} x^{2k-\alpha_0+1} dx = O(\tilde{\delta}^{\alpha_1+2\beta_1}). \quad (11)$$

Next, observing that

$$\begin{aligned} & \int_0^{C_0^{-1/\alpha_0}} (f_0(x) - f_1(x))^2 (f_0(x))^{-1} dx \\ &= \int_0^{\tilde{\delta}} (C_0 \alpha_0 x^{\alpha_0-1} - C_1 \alpha_1 x^{\alpha_1-1} - \Delta(x))^2 (C_0 \alpha_0 x^{\alpha_0-1})^{-1} dx \\ & \quad + \int_{\tilde{\delta}}^{C_0^{-1/\alpha_0}} (C_0 \alpha_0 x^{\alpha_0-1} - C_2 \alpha_0 x^{\alpha_0-1})^2 (C_0 \alpha_0 x^{\alpha_0-1})^{-1} dx \\ &\leq 2 \int_0^{\tilde{\delta}} (C_0 \alpha_0 x^{\alpha_0-1} - C_1 \alpha_1 x^{\alpha_1-1})^2 (C_0 \alpha_0 x^{\alpha_0-1})^{-1} dx \\ & \quad + 2 \int_0^{\tilde{\delta}} (\Delta(x))^2 (C_0 \alpha_0 x^{\alpha_0-1})^{-1} dx + (C_0 - C_2)^2 \int_{\tilde{\delta}}^{C_0^{-1/\alpha_0}} C_0^{-1} \alpha_0 x^{\alpha_0-1} dx, \end{aligned}$$

then, introducing (8), combining (9) and (10), and using (11), give

$$\int_0^{C_0^{-1/\alpha_0}} (f_0(x) - f_1(x))^2 (f_0(x))^{-1} dx \leq O(\tilde{\gamma}^2 \tilde{\delta}^{\alpha_1} + \tilde{\delta}^{2\beta_1+\alpha_1}).$$

(6) immediately follows taking γ and δ instead of $\tilde{\gamma}$ and $\tilde{\delta}$, as in the proof of Theorem 1 given in [4]. Considering $\tilde{\gamma}$ and $\tilde{\delta}$, we now have

$$O(\tilde{\gamma}^2 \tilde{\delta}^{\alpha_1} + \tilde{\delta}^{2\beta_1+\alpha_1}) = O(\lambda^2 n^{-2\nu-\nu\alpha_1/\beta_1} + n^{-\nu(2\beta_1+\alpha_1)/\beta_1}) = O(n^{-\nu(2\beta_1+\alpha_1)/\beta_1}),$$

and (6) then follows too since

$$\frac{\beta_1}{2\beta_1 + \alpha_1} = \frac{\beta_0}{2\beta_0 + \alpha_0} \leq \nu.$$

The result (7) will follow if we prove that

$$|C_2 \alpha_0 x^{\alpha_0-1} - C_1 \alpha_1 x^{\alpha_1-1}| \leq K x^{\alpha_1+\beta_1-1} \quad (12)$$

uniformly in $\tilde{\delta} < x \leq C_0^{-1/\alpha_0}$ and large n . By (5),

$$\begin{aligned} |C_2 \alpha_0 x^{\alpha_0-1} - C_0 \alpha_0 x^{\alpha_1-1}| &= \alpha_0 x^{\alpha_0-1} |C_2 - C_0| \\ &= \frac{C_0 C_1}{\alpha_0} \tilde{\gamma} \tilde{\delta}^{\alpha_1} \alpha_0 x^{\alpha_0-1} = K n^{-\nu} n^{-\nu(\alpha_1-\beta_1-\tilde{\gamma})/\beta_1} \tilde{\delta}^{\tilde{\gamma}+\beta_1} x^{\alpha_0-1} \leq K n^{-\nu\alpha_0/\beta_1} x^{\alpha_1+\beta_1-1} \end{aligned}$$

and so (12) will follow if we show that for $\tilde{\delta} < x \leq C_0^{-1/\alpha_0}$,

$$|C_0 \alpha_0 x^{\alpha_0-1} - C_1 \alpha_1 x^{\alpha_1-1}| \leq K x^{\alpha_1+\beta_1-1}. \quad (13)$$

But, using (3) and (4) gives, by $\tilde{\gamma} = \lambda \tilde{\delta}^{\beta_1}$, $C_0 \alpha_0 = C_0 C_1 \tilde{\delta}^{\alpha_1} (\alpha_0 - \alpha_1) + C_1 \alpha_1 \tilde{\delta}^{\tilde{\gamma}}$,

$$\begin{aligned} |C_0 \alpha_0 x^{\alpha_0-1} - C_1 \alpha_1 x^{\alpha_1-1}| &= x^{\alpha_1-1} |C_0 C_1 \tilde{\delta}^{\alpha_1} (\alpha_0 - \alpha_1) x^{-\tilde{\gamma}} + C_1 \alpha_1 \tilde{\delta}^{\tilde{\gamma}} x^{-\tilde{\gamma}} - C_1 \alpha_1| \\ &\leq K_1 x^{\alpha_1-1} \tilde{\gamma} \tilde{\delta}^{\alpha_1-\tilde{\gamma}} + K_2 x^{\alpha_1-1} |1 - (\tilde{\delta}/x)^{\tilde{\gamma}}| \\ &\leq K_3 x^{\alpha_1+\beta_1-1} \tilde{\delta}^{\alpha_1} + K_4 x^{\alpha_1-1} \tilde{\gamma} \log(x/\tilde{\delta}). \end{aligned} \quad (14)$$

Now, $x^{-\beta_1} \tilde{\gamma} \log(x/\tilde{\delta}) = (x/\tilde{\delta})^{\beta_1} \log(x/\tilde{\delta})$, and is maximized by taking $x/\tilde{\delta} = e^{1/\beta_1}$. Therefore by (14),

$$|C_0 \alpha_0 x^{\alpha_0-1} - C_1 \alpha_1 x^{\alpha_1-1}| \leq K_3 x^{\alpha_1+\beta_1-1} \tilde{\delta}^{\alpha_1} + K_5 x^{\alpha_1+\beta_1-1} \leq K_6 x^{\alpha_1+\beta_1-1}$$

uniformly in $\tilde{\delta} < x \leq C_0^{-1/\alpha_0}$. This proves (13), and completes the proof of (7).

For what follows, the proof of Theorem 1 in [4] was inspired by Farrell (1982) [3]. Observing that, using the Cauchy-Schwarz inequality,

$$\begin{aligned} P_{f_1}(|\alpha_n(X_1, \dots, X_n) - \alpha_1| \leq a_n) &= E_{f_0} \left[I(|\alpha_n(X_1, \dots, X_n) - \alpha_1| \leq a_n) \prod_{i=1}^n (f_1(X_i)/f_0(X_i)) \right] \\ &\leq (P_{f_0}(|\alpha_n(X_1, \dots, X_n) - \alpha_1| \leq a_n))^{1/2} \\ &\quad \times \left(E_{f_0} \left[\prod_{i=1}^n (f_1(X_i)/f_0(X_i))^2 \right] \right)^{1/2}, \end{aligned} \quad (15)$$

and

$$\begin{aligned} \left(E_{f_0} \left[\prod_{i=1}^n \left(\frac{f_1(X_i)}{f_0(X_i)} \right)^2 \right] \right)^{1/n} &= \int_0^{C_0^{-1/\alpha_0}} \frac{(f_1(x))^2}{f_0(x)} dx \\ &= 1 + \int_0^{C_0^{-1/\alpha_0}} (f_1(x) - f_0(x))^2 (f_0(x))^{-1} dx \\ &= 1 + O(n^{-1}), \end{aligned}$$

using (6). Hence

$$P_{f_1}(|\alpha_n(X_1, \dots, X_n) - \alpha_1| \leq a_n) \leq K (P_{f_0}(|\alpha_n(X_1, \dots, X_n) - \alpha_1| \leq a_n))^{1/2}. \quad (16)$$

By hypothesis and by (7), the left-hand side of (16) tends to 1 as $n \rightarrow \infty$. Therefore $P_{f_0}(|\alpha_n(X_1, \dots, X_n) - \alpha_1| \leq a_n)$ is bounded away from zero as $n \rightarrow \infty$. Also by hypothesis, $P_{f_0}(|\alpha_n(X_1, \dots, X_n) - \alpha_1| \leq a_n)$ tends to 1 as $n \rightarrow \infty$, and so

$$P_{f_0}(\{|\alpha_n(X_1, \dots, X_n) - \alpha_1| \leq a_n\} \cap \{|\alpha_n(X_1, \dots, X_n) - \alpha_0| \leq a_n\})$$

is bounded away from zero. Consequently, for large n ,

$$|\alpha_1 - \alpha_0| \leq 2a_n$$

that is, $\tilde{\gamma} = \lambda n^{-\nu} \leq 2a_n$, and so

$$\lim_{n \rightarrow \infty} n^\nu a_n \geq \frac{\lambda}{2}.$$

Since this is true for each $\lambda > 0$, Theorem 1 is proved.

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